A-PDF MERGER DEMO

Analysis IV

Mid-term Examination

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{\frac{1}{2}x^2}$ and $g(x) = e^{-\frac{1}{2}x^2}$. Show that g is uniformly continuous whereas f is not.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = e^{-\frac{1}{2}|x|^2}$. Construct a sequence g_m of functions which are continuous and compactly supported such that $g_m \to f$ uniformly on $\mathbb{R}^n$ as $m \to \infty$.

3. Let $B_r(0)$ be the ball in $\mathbb{R}^n$ centred at 0 with radius r. Construct a radial function which is continuous, identically 1 on $B_r(0)$ and identically 0 outside $B_{2r}(0)$. Find also such a non-radial function.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0) = 0$ and

$$f(x,y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4+y^2)^2}, \quad (x,y) \neq 0.$$

Show that f is continuous by proving and using the inequality

$$4x^4y^2 \leq (x^4+y^2)^2.$$

5. For $t \in \mathbb{R}$ define $f(t) = t + 2t^2 \sin\frac{1}{t}$, $t \neq 0$ and $f(0) = 0$. Show that f is not one to one in any nhood of 0 and explain why we cannot apply inverse function theorem even though $f'(0) \neq 0$.

6. Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$ is not one to one but every point $w \in \mathbb{C}$ has a nhood in which f is one to one.

7. Let f_n be a uniformly bounded sequence of Riemann integrable functions on $[a, b]$ and define

$$F_n(x) = \int_a^x f_n(t)\,dt, \qquad a \leq x \leq b.$$

Prove that there exists a subsequence F_{n_k} which converges uniformly on $[a, b]$.

8. Let X be a compact metric space and let $f : X \to X$ be an isometry. Show that f is onto.

9. Prove Weierstrass approximation theorem for functions in $C[0, \frac{1}{2}]$ by using Fejer's theorem for $C(S^1)$ where S^1 is the circle group.

10. Let X be a compact Hausdorff space. Let $A \subset C(X, \mathbb{R})$ be an algebra with the property that given any $x_0 \in X$ $\exists$ $f \in A$ such that $f(x_0) \neq 0$. Show that $\exists$ a function $g \in A$ such that $g(x) \geq \delta > 0$ for all $x \in X$.

11. Let $f \in C(S^1)$. By considering the integrals

$$\int_0^1 \left(f(x+\tfrac{1}{2N}) - f(x)\right) e^{-2\pi i n N x} \, dx$$

or otherwise show that $\hat{f}(n) \to 0$ as $n \to \pm\infty$ where

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx.$$

12. Let $D_n(x) = \dfrac{\sin(n+\frac{1}{2})2\pi x}{\sin \pi x}$ be the Dirichlet kernel for which we know $\int_0^1 D_n(x)\,dx = 1$. Using the result of Problem 11 and the fact that the function

$$g(x) = \left(\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right)$$

is continuous, show that

$$\lim_{N\to\infty} \int_0^N \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$